CYCLIC DISPLACEMENTS FOR THE GENERALIZED AREA INTEGRAL

(TSIKLICHESKIE PEREMESHCHENIIA DLIA OBOBSHCHENNOGO INTEGRALA PLOSHCHADEI)

PMM Vol.25, No.4, 1961, pp. 774-777

A.A. BOGOIAVLENSKII
(Moscow)

(Received May 8, 1961)

Chaplygin in [1] generalized the area theorem and derived the appropriate integrals. The possibility of generalizing the results, in the Chaplygin sense, was also indicated. The most typical generalized area integrals are given in Sections 1, 2, 4 of [1].

Without dealing with all the generalizations which follow from [1], one can still show that the given integrals are in essence integrals of cyclic displacements according to Chetaev [2].

We retain the notation and the definitions adopted by Chaplygin.

Regarding Section 1 of [1]. The properties of constraints which are imposed on a system are revealed by means of those possible displacements which consist of the rotation of a system of mass points, without altering their configuration, about a straight line Az.

Let the angle $\delta\theta$ represent the rotation.

The position of the mechanical system can be determined by Poincaré-Chetaev dependent variables

$$\alpha, \quad \beta, \quad \gamma, \quad \beta_i^k, \quad x', \quad y', \quad z' \quad (i, k = 1, 2) \tag{1}$$

where β_i^k are the cosines of the angles between the $Axyz$-system and the newly introduced rectangular system $Ax'y'z'$, whose z'-axis coincides with the straight line Az, and the system rotates about this axis through angle $\delta\theta$. The lower subscript relates to the x, y, z-axes, the superscript to the x', y', z'-axes.

The variables β_i^k satisfy the following equations:

$$\frac{d\beta_i^1}{dt} = \beta_i^2 \frac{d\theta}{dt}, \qquad \frac{d\beta_i^2}{dt} = -\beta_i^1 \frac{d\theta}{dt} \qquad (i, k = 1, 2) \tag{2}$$

Real displacements are determined by the independent variables η_i for which we can take

$$\frac{d\alpha}{dt}, \quad \frac{d\beta}{dt}, \quad \frac{d\gamma}{dt}, \quad \frac{d\theta}{dt}, \quad \eta_\nu \qquad (\nu = 5, 6, \ldots)$$

Here η_ν is a part of the velocity component of the mass points in the moving coordinate system $x'y'z'$ on these axes, or appropriately chosen independent variables representing the motion of the mechanical system in this coordinate system. A change in the arbitrary function f due to variables (1) for a given possible displacement can be determined by

$$\delta f = \sum_{j=1}^{4} \omega_j X_j f + \Sigma \omega_\nu X_\nu f$$

The parameters of possible displacements are

$$\omega_1 = \delta\alpha, \quad \omega_2 = \delta\beta, \quad \omega_3 = \delta\gamma, \quad \omega_4 = \delta\theta, \quad \omega_\nu \quad (\nu = 5, 6, \ldots)$$

The operators for the respective displacements are

$$X_1 = \frac{\partial}{\partial\alpha}, \quad X_2 = \frac{\partial}{\partial\beta}, \quad X_3 = \frac{\partial}{\partial\gamma}, \quad X_4 = \sum_{i=1}^{2}\left(\beta_i^2 \frac{\partial}{\partial\beta_i^1} - \beta_i^1 \frac{\partial}{\partial\beta_i^2}\right), \quad X_{\nu\,(\nu=5,\,6\,\ldots)}$$

For the parameters ω_ν of possible displacements one can take infinitely small changes in the relative coordinates of the mass points in the $x'y'z'$-system.

Then the form of the operators X_ν, similar to $X_i (i = 1, 2, 3)$, becomes obvious. The operators X_ν are independent of variables α, β, γ, β_i^k and constitute a sub-group of relative displacements. All possible displacements constitute an Abel group.

With corresponding [or appropriate] constraints it is possible to choose the parameters ω_ν so that the number of them is no less than the total number of relative coordinates. This should be borne in mind in all the sections.

From the expression for the kinetic energy T of the system it is evident that

$$X_4 (T) = 0$$

The forces applied to the system are such that the points of application of two derived forces can be chosen independently of β_i^k. We then have

$$X_4 (U) = 0$$

Displacement X_4 is indeed a cyclic Chetaev displacement. The first

Chaplygin integral (2) from [1] corresponds to this displacement.

$$\frac{\partial L}{\partial \eta_4} = \text{const}$$

Regarding Section 2 of [1]. As in Section 1, let the angle $\delta\theta$ represent rotation of part I and $\delta\phi$ that of part II of the system.

The position of the mechanical system can be determined from the dependent variables

$$\alpha,\ \beta,\ \gamma,\ \alpha',\ \beta',\ \gamma',\ \beta_i^k,\ \alpha_i^k,\ x_1',\ y_1',\ z_1',\ x_2',\ y_2',\ z_2' \tag{3}$$

where β_i^k, α_i^k are cosines of the axes angles for parts I and II of the system, respectively, x_1', y_1', z_1', x_2', y_2', z_2' are the relative coordinates of mass points in the corresponding systems of axes, as derived for Section 1.

The variables β_i^k satisfy Equations (2), α_i^k satisfy the same ones after replacing α_i^k, ϕ by β_i^k, θ.

Possible displacements of the system will be determined by the relations

$$\delta\varphi = k\delta\theta, \qquad (k = L : L')$$

Real displacements are represented by the variables

$$\frac{d\alpha}{dt},\ \frac{d\beta}{dt},\ \frac{d\gamma}{dt},\ \frac{d\alpha'}{dt},\ \frac{d\beta'}{dt},\ \frac{d\gamma'}{dt},\ \frac{d\theta}{dt},\quad \eta_\nu \quad (\nu=8,\ 9,\ \ldots)$$

Quantities η_ν have the same significance as in Section 1 for parts I and II of the system.

The rotations $d\theta/dt$ and $d\phi/dt$ of these parts of the system are connected by the same expressions, as are the possible displacements

$$\frac{d\varphi}{dt} = k\,\frac{d\theta}{dt}$$

A change in the arbitrary function f due to variables (3) over a possible displacement is

$$\delta f = \sum_{j=1}^{7} \omega_j X_j f + \Sigma \omega_\nu X_\nu f$$

The parameters of possible displacements are

$$\omega_1 = \delta\alpha,\ \omega_2 = \delta\beta,\ \omega_3 = \delta\gamma,\ \omega_4 = \delta\alpha',\ \omega_5 = \delta\beta',\ \omega_6 = \delta\gamma',\ \omega_7 = \delta\theta,\ \omega_\nu \quad (\nu=8,\ 9,\ \ldots)$$

The operators for the respective displacements are

$$X_1 = \frac{\partial}{\partial\alpha}, \quad X_2 = \frac{\partial}{\partial\beta}, \quad X_3 = \frac{\partial}{\partial\gamma}, \quad X_4 = \frac{\partial}{\partial\alpha'}, \quad X_5 = \frac{\partial}{\partial\beta'}, \quad X_6 = \frac{\partial}{\partial\gamma'}$$

$$X_7 = \sum_{i=1}^{2}\left\{\left(\beta_i^2\frac{\partial}{\partial\beta_i^1} - \beta_i^1\frac{\partial}{\partial\beta_i^2}\right) + k\left(\alpha_i^2\frac{\partial}{\partial\alpha_i^1} - \alpha_i^1\frac{\partial}{\partial\alpha_i^2}\right)\right\}, \quad X_\nu \quad (\nu=8,\,9,\,\ldots)$$

For parameters ω_ν of possible displacements, infinitely small changes in mass point coordinates in the corresponding systems of axes can be taken. The form of operator X_ν is obvious. These operators are independent of variables α, β, γ, α', β', γ', β_i^k, α_i^k, and they constitute a sub-group of relative displacements of the mechanical system.

From the kinetic energy expression and potential function it is evident that

$$X_7(T + U) = 0$$

Displacement X_7 is a cyclic one in the sense of Chetaev.

The first Chaplygin integral of [1] corresponds to this displacement

$$S + kS' = \text{const}$$

These results can be easily extended to the system dealt with by Chaplygin in Section 3 of [1].

Regarding Section 4 of [1]. The results of this section have been further applied by the author [3], in the sense that the given integral is found for different forces and constraints applied to the mechanical system. The cyclic displacements will therefore be found under these new conditions. The analysis is retained for the Chaplygin condition.

Suppose the angle $\delta\theta$ represents the rotation of system I as in Section 1, while to translation along straight line n of system II there corresponds a displacement through distance δl, assuming invariance of systems I and II.

We use the same notation as in Sections 1 and 2.

The position of the mechanical system is determined by the dependent variables

$$\alpha,\ \beta,\ \gamma,\ \beta_i^k,\ x_1',\ y_1',\ z_1',\ x_2,\ y_2,\ z_2 \tag{4}$$

where x_2, y_2, z_2 are mass point coordinates of system II in $Axyz$-coordinates.

Possible system displacements are determined by relations

$$\delta l = \varkappa\delta\theta, \qquad \varkappa^2 = a^2 + b^2$$

Let the parameters of the real displacements be

$$\frac{d\alpha}{dt}, \quad \frac{d\beta}{dt}, \quad \frac{d\gamma}{dt}, \quad \frac{d\theta}{dt}, \quad \eta_\nu \qquad (\nu=5, 6, \ldots)$$

The rotation of system I and translation of system II are connected by the same relations as the possible displacements

$$\frac{dl}{dt} = \varkappa \frac{d\theta}{dt}, \qquad \varkappa^2 = a^2 + b^2 \tag{5}$$

Variables β_i^k satisfy Equations (2), while variables x_2, y_2, on the strength of (5), satisfy

$$\frac{dx_2}{dt} = -b\frac{d\theta}{dt} + \frac{a}{\varkappa}\frac{dh}{dt}, \qquad \frac{dy_2}{dt} = a\frac{d\theta}{dt} + \frac{b}{\varkappa}\frac{dh}{dt}$$

Here dh/dt is the rate of change of the projection, on the straight line *AC*, of the distance between *A* and a mass point of system II.

The change in the position function of the system f in dependence on the variables (4) in a possible displacement is

$$\delta f = \sum_{j=1}^{4} \omega_j X_j f + \Sigma \omega_\nu X_\nu f$$

The parameters of possible displacements are

$$\omega_1 = \delta\alpha, \quad \omega_2 = \delta\beta, \quad \omega_3 = \delta\gamma, \quad \omega_4 = \delta\theta, \ \omega_\nu \qquad (\nu=5, 6, \ldots)$$

The operators of the corresponding displacements are

$$X_1 = \frac{\partial}{\partial\alpha}, \quad X_2 = \frac{\partial}{\partial\beta}, \quad X_3 = \frac{\partial}{\partial\gamma}$$

$$X_4 = \sum_{i=1}^{2}\left(\beta_i^2 \frac{\partial}{\partial\beta_i^1} - \beta_i^1 \frac{\partial}{\partial\beta_i^2}\right) + a\Sigma\frac{\partial}{\partial y_2} - b\Sigma\frac{\partial}{\partial x_2}, \qquad X_\nu \qquad (\nu=5, 6, \ldots)$$

For the parameters ω_ν of the possible displacements one can take infinitely small changes in relative coordinates of mass points of system (I), as in Sections 1 and 2, and the projections of δh on the straight line *AC* of the distance from *A* to mass points of system II.

The form of the operators X_ν for system I is obvious, while for system II it is

$$X_\nu = \frac{1}{\varkappa}\left(a\frac{\partial}{\partial x_2} + b\frac{\partial}{\partial y_2}\right)$$

These operators are independent of variables α, β, γ, β_i^k, and constitute a sub-group of relative displacements of systems I and II.

The possible displacements form an Abel group.

It follows from the kinetic energy and potential function expressions for the system

$$X_4 (T + U) = 0$$

X_4 is a Chetaev cyclic function. The first Chaplygin integral (11) of [1], or (5) from [3], corresponds to

$$\frac{\partial L}{\partial \eta_4} = \text{const}$$

The expressions obtained here are applicable to the possibilities of generalizing the area theorem pointed out by Chaplygin.

Incidentally, we make one observation of Section 9 of [1].

According to the area theorem, the derivatives of the sum of the moments of momentum of the motion of the envelope are written down as [1, p. 52]

$$\frac{d}{dt}(J_x p - Mav) = -aN_y, \qquad \frac{d}{dt}(J_y q + Mau) = aN_x$$

This is inaccurate. The second equation should have been as follows:

$$\frac{d}{dt}(J_y q + Mau) = aN_x + aMg \sin \varphi$$

Integrals obtained from this equation should be altered accordingly.

Integration of the differential equations deduced by Chaplygin by quadratures, however, can only be carried out for the case $\phi = 0$, i.e. when all the terms left out are eliminated. This reservation does not influence the final form of the formulas.

BIBLIOGRAPHY

1. Chaplygin, S.A., O nekotorom vozmozhnom obobshchenii teoremy ploshchadei s prilozheniem k zadache o katanii sharov (Possible generalization of the area theorem as applied to the problem of rolling spheres). *Sobr. Soch. (Collected Works)*, Vol. 1, pp. 26-58. Gostekhizdat, 1948.

2. Chetaev, N.G., Ob uravneniiakh Puankare (Poincaré's equations). *PMM* Vol. 5, No. 2, 1941.

3. Bogoiavlenskii, A.A., Ob odnom vide obobshchennogo integrala ploshchadei (On a form of the generalized area integrals). *PMM* Vol. 21, No. 3, 1957.

Translated by V.H.B.